Approximate Transmission Conditions for the Laplacian in a High Contrast Medium with a Thin Layer. The Influence of the Curvature.

Ronan Perrussel, Clair Poignard

Approximate Transmission Conditions for the Laplacian in a High Contrast Medium with a Thin Layer. The Influence of the Curvature.

R. Perrussel[1], **C. Poignard**[2,*]

[1] Laboratoire Ampère, UMR CNRS 5005 & Ecole Centrale de Lyon & Université de Lyon
[2] INRIA Bordeaux-Sud-Ouest & IMB UMR CNRS 5251. 351, cours de la Libération, 33405 Talence.
*Email: clair.poignard@inria.fr

Abstract

We study the steady state potential in high contrast media with resistive or conductive thin layer. In both cases equivalent transmission conditions are given and the influence of the curvature is explicitly notified unlike the weak contrast media.

Introduction

We present approximate transmission conditions for the electroquasistatic potential in a high contrast medium with a thin layer. We consider two kinds of thin layers: the first one is resistive with a conductivity of the same order as its thickness, whereas the second one has its resistivity of the order of the thickness. Our goal is to show the influence of the curvature of the layer. The asymptotics for the thickness tending to zero are very different in the two cases, at the difference of the small diameter inhomogeneities [6].

The studied problem

Denote by Ω a bounded domain with a smooth boundary $\partial\Omega$. The domain $\mathcal{O}_i$ is a subdomain of Ω with smooth boundary Γ. It is surrounded by a thin layer $\mathcal{O}_m^\delta$ with thickness δ such that $\mathcal{O}_m^\delta \cup \mathcal{O}_i$ is compactly embedded in Ω. We denote by $\mathcal{O}_e^\delta = \Omega \setminus \overline{\mathcal{O}_m^\delta \cup \mathcal{O}_i}$. Let σ_i be the inner complex permittivity of $\mathcal{O}_i$ and we define similarly σ_e and σ_m. We define $\mathcal{O}_e = \Omega \setminus \overline{\mathcal{O}_i}$ and

$$\sigma = \begin{cases} \sigma_i, \text{in } \mathcal{O}_i, \\ \sigma_m, \text{in } \mathcal{O}_m^\delta, \\ \sigma_e, \text{in } \mathcal{O}_e^\delta, \end{cases} \quad \text{and } \tilde{\sigma} = \begin{cases} \sigma_i, \text{in } \mathcal{O}_i, \\ \sigma_e, \text{in } \mathcal{O}_e. \end{cases}$$

The quasistatic formulation is given by:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0, \text{ in } \Omega \\ u|_{\partial\Omega} = \phi, \text{ on } \partial\Omega; \end{cases} \tag{1}$$

ϕ is the electric potential imposed on the boundary of Ω. We suppose that ϕ is as regular as we need. The potential u is approximated by $u_0 + \delta u^1$, where u^0 and u^1 satisfy elementary problems in the δ-independent domain $\mathcal{O}_e \cup \mathcal{O}_i$ with appropriate transmission conditions on Γ. The error performed by this approximation is estimated similarly to [4]: for any domain Υ either compactly embedded in $\mathcal{O}_e$ or equal to $\mathcal{O}_i$ there exists a δ–independent constant $C > 0$ satisfying

$$\|u - u^0 - \delta u^1\|_{H^1(\Upsilon)} \leq C\delta^2.$$

1 Method

The method used to derive the effective transmission conditions has been extensively described previously. We refer to [3] for the original paper and to [5], [4] for a more general description. It consists in a suitable change of variable in the thin layer in order to make appear the small parameter in the equations.

1.1 Geometry

Let $\mathbf{x_T} = (x_1, x_2)$ be a system of local coordinates on $\Gamma = \{\psi(\mathbf{x_T})\}$. Denote by n the normal vector of Γ outwardly directed to $\mathcal{O}_i$ and define the map Φ by

$$\forall (\mathbf{x_T}, x_3) \in \Gamma \times \mathbb{R}, \quad \Phi(\mathbf{x_T}, x_3) = \psi(\mathbf{x_T}) + x_3 n(\mathbf{x_T}).$$

The thin layer $\mathcal{O}_m^\delta$ is then parameterized by

$$\mathcal{O}_m^\delta = \{\Phi(\mathbf{x_T}, x_3), \quad (\mathbf{x_T}, x_3) \in \Gamma \times (0, \delta)\}.$$

The Euclidean metric in $(\mathbf{x_T}, x_3)$ is given by the 3×3–matrix $(g_{ij})_{i,j=1,2,3}$ where $g_{ij} = \langle \partial_i \Phi, \partial_j \Phi \rangle$. Denote by (g^{ij}) the inverse matrix of (g_{ij}), and by g the determinant of (g_{ij}). According to [1], $g_{33} = 1$ and for $(\alpha, \beta) \in \{1, 2\}^2$

$$g_{\alpha\beta}(\mathbf{x_T}, x_3) = g^0_{\alpha\beta}(\mathbf{x_T}) - 2x_3 b_{\alpha\beta}(\mathbf{x_T}) + x_3^2 c_{\alpha\beta}(\mathbf{x_T}).$$

The mean curvature of Γ equals $\mathscr{H} = -\frac{1}{2} \left.\frac{\partial_3(\sqrt{g})}{\sqrt{g}}\right|_{x_3=0}$.

1.2 Laplace operator for functions

Laplace-Beltrami operator on functions in the metric $(g_{ij})_{i,j=1,2,3}$ is given by the well-known expression [2]

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1,2,3} \partial_i \left(\sqrt{g} g^{ij} \partial_j \right), \tag{2}$$

while Laplace-Beltrami operator on Γ equals

$$\Delta_\Gamma = \frac{1}{\sqrt{g}|_{x_3=0}} \sum_{\alpha,\beta=1,2} \partial_\alpha \Big(\left(\sqrt{g} g^{\alpha\beta} \right)\Big|_{x_3=0} \partial_\beta \Big).$$

We define the differential operator R_1 on Γ by

$$R_1 \bullet = \partial_3 \left(\frac{1}{\sqrt{g}} \partial_\alpha \left(\sqrt{g} g^{\alpha\beta} \partial_\beta \bullet \right) \right)\Big|_{x_3=0}$$

Performing the rescaling $\eta = x_3/\delta$, $\mathcal{O}_m^\delta$ is diffeomorphic to the cylinder $\mathbb{T} = \Gamma \times (0, 1)$. We also infer

$$\Delta_g = \frac{1}{\delta^2} \partial_\eta^2 - \frac{2}{\delta} \mathscr{H} \partial_\eta + \Delta_\Gamma + \eta \partial_3 \left(\frac{\partial_3(\sqrt{g})}{\sqrt{g}} \right)\Big|_{x_3=0} \partial_\eta$$
$$+ \delta \eta^2 \partial_3^2 \left(\frac{\partial_3(\sqrt{g})}{\sqrt{g}} \right)\Big|_{x_3=0} \partial_\eta + \delta R_1 + \delta^2 R_\delta,$$

where R_δ is a second order differential operator on Γ. There exists $C > 0$ such that for $s \geq 0$

$$\forall u \in H^{s+1}(\mathbb{T}),\ \|R_\delta u\|_{H^{s-1}(\Gamma\times[0,1])} \leq C\|u\|_{H^{s+1}(\Gamma\times[0,1])}.$$

Denote by $u_m = u \circ \Phi$. The transmission conditions inherent to (1) write now:

$$\begin{cases} \sigma_c \partial_n u|_{\Gamma^-} = (\sigma_m/\delta)\partial_\eta u_m|_{\eta=0},\ u|_{\Gamma^-} = u_m|_{\eta=0}, \\ \sigma_e \partial_n u|_{\Gamma^+_\delta} = (\sigma_m/\delta)\partial_\eta u_m|_{\eta=1},\ u|_{\Gamma^+_\delta} = u_m|_{\eta=1}. \end{cases}$$

To obtain our transmission conditions, we set

$$u = u^0 + \delta u^1 + \cdots, \text{in } \mathcal{O}_e^\delta \cup \mathcal{O}_i,\ u_m = u_m^0 + \delta u_m^1 + \cdots, \text{in } \mathbb{T}$$

and we perform the identification of the terms with the same power in δ. We emphasize that the coefficients u^j will be defined in the whole domain Ω even if $u^0 + \delta u^1$ approximates u only in $\mathcal{O}_e^\delta \cup \mathcal{O}_i$. We necessarily have:

$$\text{For } j = 0, 1,\ \Delta u^j = 0, \text{in } \mathcal{O}_e \cup \mathcal{O}_i, \quad u^j|_{\partial\Omega} = \delta_0^j \phi. \quad (3)$$

The influence of the layer is confined in the transmission conditions. For weak contrast media, the geometry does not appear in the first order approximation [4]. We present here the influence of the curvature when the layer is highly resistive or highly conducting.

2 Approximate transmission conditions

In the following, α_m denotes a complex number such that $|\alpha_m|$, $1/|\alpha_m| = O(1)$.

2.1 Resistive thin layer: $\sigma_m = \alpha_m \delta$.

- The order 0.

$$\left[\widetilde{\sigma}\partial_n u^0\right]_\Gamma = 0, \quad \left[u^0\right]_\Gamma - \frac{\sigma_c}{\alpha_m}\partial_n u^0|_{\Gamma^-} = 0.$$

- The order 1.

$$\left[\widetilde{\sigma}\partial_n u^1\right]_\Gamma = \sigma_e \Delta_\Gamma u^0|_{\Gamma^+},$$
$$\left[u^1\right]_\Gamma - \frac{\sigma_c}{\alpha_m}\partial_n u^1|_{\Gamma^-} = -\left(1 - \mathscr{H}\frac{\sigma_e}{\alpha_m}\right)\partial_n u^0|_{\Gamma^+}.$$

2.2 Conducting thin layer: $\sigma_m = \alpha_m/\delta$.

- The order 0.

$$\left[\widetilde{\sigma}\partial_n u^0\right]_\Gamma + \alpha_m \Delta_\Gamma u^0|_\Gamma = 0, \quad \left[u^0\right]_\Gamma = 0.$$

- The order 1.

$$\left[\widetilde{\sigma}\partial_n u^1\right]_\Gamma + \alpha_m \Delta_\Gamma u^1|_{\Gamma^-} = (\sigma_e + \alpha_m \mathscr{H})\Delta_\Gamma u^0|_\Gamma$$
$$- \alpha_m R_1 u^0|_\Gamma,$$
$$\left[u^1\right]_\Gamma = -\partial_n u^0|_{\Gamma^+}.$$

2.3 Numerical simulations for a 2D–resistive thin layer

Domains Ω and $\mathcal{O}_i$ are two disks centered in 0 of respective radius equal to 2 and 1. Parameters σ_c and α_m are taken equal to 1 and α_e to 2. Function ϕ is $\exp(\mathrm{i}3\theta)$. From the analytic solutions the expected rates of convergence with δ are verified since the numerical estimated exponents are respectively 1 for the order 0 and 1.97 for the order 1; see Figure 1.

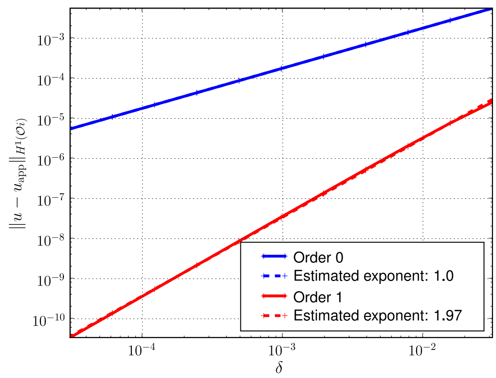

Figure 1: Log-log diagram of the error versus the membrane thickness.

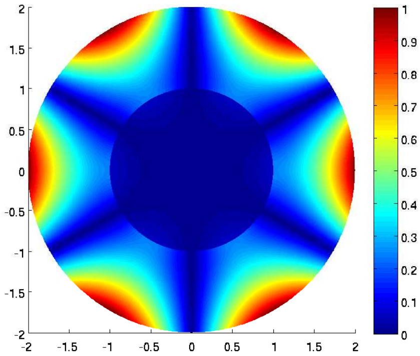

Figure 2: Real part of $u^0 + \delta u^1$ computed by the finite element method with $\delta = 0.05$

References

[1] M.P. do Carmo. *Differential geometry of curves and surfaces*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1976. Translated from the Portuguese.

[2] B.A. Dubrovin, A.T. Fomenko, and S.P. Novikov. *Modern geometry—methods and applications. Part I*, volume 93 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992. The geometry of surfaces, transformation groups, and fields, Translated from the Russian by Robert G. Burns.

[3] B. Engquist and J.C. Nédélec. Effective boundary condition for acoustic and electromagnetic scattering thin layer. *Technical Report of CMAP*, 278, 1993.

[4] C. Poignard. Approximate transmission conditions through a weakly oscillating thin layer. *Math. Meth. App. Sci.*, 32:435–453, 2009.

[5] C. Poignard, P. Dular, R. Perrussel, L. Krähenbühl, L. Nicolas, and M. Schatzman. Approximate conditions replacing thin layer. *IEEE Trans. on Mag.*, 44(6):1154–1157, 2008.

[6] M.S. Vogelius and H-M. Nguyen. A representation formula for the voltage perturbations caused by diametrically small conductivity inhomogeneities. Proof of uniform validity. Submitted.